

Renormalization Group and Infinite Algebraic Structure in D -Dimensional Conformal Field Theory.

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Abstract. We consider scalar field theory in the D -dimensional space with nontrivial metric and local action functional of most general form. It is possible to construct for this model a generalization of renormalization procedure and RG-equations. In the fixed point the diffeomorphism and Weyl transformations generate an infinite algebraic structure of D -Dimensional conformal field theory models. The Wilson expansion and crossing symmetry enable to obtain sum rules for dimensions of composite operators and Wilson coefficients.

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1. Introduction

Essential achievements in investigations of quantum field theoretical models were obtained on the basis of analysis of their symmetry properties and algebraic structures. The higher is the symmetry, the more restrictions put it on the possible form of correlation functions. It is known that the conformal invariance defines two point correlation functions up to a constant amplitude and three point ones as a finite linear combination of known functions (Polyakov triangles) with constant coefficients [1],[2],[3]. In studies of 2-dimensional conformal field theory it was found that the Virasoro algebra described its most fundamental features [4],[5]. An approach to the extension of these methods on conformal field theory in D dimensions was suggested in [6] for any D . In this papers, it was proposed to use the algebra of the general coordinate transformation as an analog of the Virasoro algebra for the D -dimensional case. In [6] the Green functions for operators ϕ^2 , ϕ^4 were studied in the theory ϕ^4 . For a generalized diffeomorphism combined from diffeomorphism and Weyl transformations the Ward identities for these Green functions were obtained which are similar to ones used in 2-dimension conformal theory. In this paper we generalize the results of [6] for all the composite operators of the scalar D -dimensional conformal field theory and construct the infinite algebraic structure analogous to one presented by Virasoro algebra in two

dimensions. For this purpose we analyze the diffeomorphism and Weyl transformation in D -dimensional curved space and the most essential features of the renormalization procedure in quantum field theory [7]. We use the Wilson operator product expansion and crossing symmetry for construction of sum rules for critical exponents and Wilson coefficients in conformal field theory [8].

2. Diffeomorphism transformations

For the curved D -dimensional space with metric $\gamma_{\mu\nu}$, the general coordinate (diffeomorphism) transformations are defined in the following way. The infinitesimal reparametrization of the coordinates x is written as : $\delta_\alpha^{DT} x^\mu = \alpha^\mu(x)$, where $\alpha_\mu(x)$ are the parameters of transformation. The commutation relation for diffeomorphism transformations (DT) is of the form:

$$[\delta_\alpha^{DT}, \delta_\beta^{DT}] = \delta_{[\alpha, \beta]}^{DT}, \quad (1)$$

where $[\alpha, \beta] = (\alpha \nabla) \beta - (\beta \nabla) \alpha$ is the commutator of vector fields (∇_λ denotes the covariant derivative, $\nabla_\lambda \gamma_{\mu\nu} = 0$). For tensor fields,

$$\delta_\alpha^{DT} F(x) = L_\alpha F(x) \quad (2)$$

Here L_α denotes the Lie derivative defined by the vector field $\alpha^\mu(x)$:

$$L_\alpha F_{\nu_1, \dots, \nu_n}^{\mu_1, \dots, \mu_m}(x) = (\alpha \nabla) F_{\nu_1, \dots, \nu_n}^{\mu_1, \dots, \mu_m} + \sum_{i=1}^n \nabla_{\nu_i} \alpha^{\lambda_i} F_{\nu_1, \dots, \lambda_i, \dots, \nu_n}^{\mu_1, \dots, \mu_m} - \sum_{i=1}^m \nabla_{\lambda_i} \alpha^{\mu_i} F_{\nu_1, \dots, \nu_n}^{\mu_1, \dots, \lambda_i, \dots, \mu_m}.$$

Particularly, for the scalar field ϕ and the metric $\gamma_{\mu\nu}(x)$

$$\delta_\alpha^{DT} \phi(x) = (\alpha \nabla) \phi(x), \quad \delta_\alpha^{DT} \gamma_{\mu\nu} = \nabla_\mu \alpha_\nu + \nabla_\nu \alpha_\mu \quad (3)$$

Let us introduce the notation:

$$\omega_\alpha^{\mu\nu}(\gamma) \equiv \nabla^\mu \alpha^\nu + \nabla^\nu \alpha^\mu - \frac{2}{D} (\nabla \alpha) \gamma^{\mu\nu}.$$

The vector $\alpha(x)$ for which $\omega_\alpha^{\mu\nu} = 0$ and the corresponding transformation $\delta_\alpha^{conf} \equiv \delta_\alpha^{DT}$ will be called conformal ones. It is well known that in the flat space $\delta_\alpha^{conf} x$ is the conformal transformation (CT) of the coordinates x . The commutator $[\alpha, \beta]$ of conformal vectors α, β is conformal. Therefore it follows from (1) that the CTs form a subgroup of the DT group. This subgroup will be called conformal.

Let $\Phi(x)_{\mu_1, \dots, \mu_n}$ denotes the covariant tensor, obtained by multiplications of the covariant derivatives of the field ϕ and the curvature tensors with possible contraction of part of the indices. The set $\{\Phi\}$ of all such tensors can be used as the basis for constructing diffeomorphism invariant local functionals of the field ϕ . Thus the most general form of the diffeomorphism invariant local action of the field ϕ in the curved D -dimensional space can be written as follows [3,4]:

$$S(A, \gamma, \phi) = \int dx \sqrt{\gamma} L(\phi(x), A(x)),$$

where $\gamma \equiv \det \gamma_{\mu\nu}$, $L(\phi(x), A(x))$ is the Lagrangian:

$$L(\phi(x), A(x), \gamma(x)) = \sum_{\Phi_i \in \{\Phi\}} A^i(x) \Phi_i(x)$$

and $A^i(x)$ denotes the contravariant tensor source corresponding to the covariant tensor field $\Phi_i(x)$. In the Lagrangian the indices of sources and the fields are contracted.

The generating functional for the connected Green functions of the Euclidean quantum field theory with action S has the form:

$$W(A, \gamma) = \ln \int \exp\{-S(\phi, A)\} D\phi$$

The metric $\gamma_{\mu\nu}(x)$ can be considered as the source for the energy-momentum tensor.

Taking into account the diffeomorphism invariance of the action $S(\phi, A)$ and using the Schwinger equations for W it is easy to show that the functional $W(A, \gamma)$ is invariant in respect to the DTs:

$$\delta_\alpha^{DT} W(A, \gamma) = D_\alpha^{DT} W(A, \gamma) = 0. \quad (4)$$

We have used the notation:

$$D_\alpha^{DT}(A, \gamma) \equiv \delta_\alpha^{DT} \gamma_{\mu\nu} \frac{\delta}{\delta \gamma_{\mu\nu}} + \sum_i \delta_\alpha^{DT} A^i \frac{\delta}{\delta A^i}$$

where $\delta_\alpha^{DT} \gamma_{\mu\nu}$, $\delta_\alpha^{DT} A^i$ are defined by (2), (3). Obviously, the operators $D_\alpha^{DT}(A, \gamma)$ form a representation of the diffeomorphism algebra:

$$[D_\alpha^{DT}(A, \gamma), D_\beta^{DT}(A, \gamma)] = D_{[\alpha, \beta]}^{DT}(A, \gamma).$$

3. Weyl transformations

We consider now the group of the Weyl transformations (WT). For the metric $\gamma_{\mu\nu}$, the infinitesimal WT is defined as the local rescaling

$$\delta_\sigma^W \gamma_{\mu\nu}(x) = -2\sigma(x) \gamma_{\mu\nu}(x) \quad (5)$$

specified by the scalar function $\sigma(x)$.

These transformations form the commutative algebra:

$$[\delta_\sigma^W, \delta_\rho^W] = 0. \quad (6)$$

For the field ϕ we define the WT in the following way:

$$\delta_\sigma^W \phi(x) = \sigma(x) d_\phi \phi(x), \quad (7)$$

where $d_\phi = (D - 2)/2$ is the canonical dimension of the field ϕ . The definitions (5), (7) make it possible to define the WT for the set of fields Φ :

$$\delta_\sigma^W \Phi(x) \equiv \left(\delta_\sigma^W \gamma_{\mu\nu} \frac{\delta}{\delta \gamma_{\mu\nu}} + \delta_\sigma^W \phi \frac{\delta}{\delta \phi} \right) \Phi(x).$$

This transformation can be written in the form:

$$\delta_\sigma^W \Phi_i(x) = \sum_j M_i^j(\sigma) \Phi_j(x)$$

with the matrix $M_i^j(\sigma) = M_i^j(\sigma, \gamma)$ satisfying the relation:

$$\delta_\sigma^W \gamma_{\mu\nu} \frac{\delta}{\delta \gamma_{\mu\nu}} M(\rho) - \delta_\rho^W \gamma_{\mu\nu} \frac{\delta}{\delta \gamma_{\mu\nu}} M(\sigma) + [M(\sigma), M(\rho)] = 0. \quad (8)$$

Let us define the operator

$$D_\sigma^W(A, \gamma, \phi) \equiv \delta_\sigma^W \gamma_{\mu\nu} \frac{\delta}{\delta \gamma_{\mu\nu}} + \delta_\sigma^W \phi \frac{\delta}{\delta \phi} + \sum_i \delta_\sigma^W A^i \frac{\delta}{\delta A^i},$$

were

$$\delta_\sigma^W A^i \equiv \sum_j \left(\sigma D \delta_j^i - M_j^i(\sigma) \right) A^j. \quad (9)$$

It can be considered as a general form of infinitesimal WT suitable for all fields and sources because from (8), (9) it follows that

$$[D_\sigma^W(A, \gamma, \phi), D_\rho^W(A, \gamma, \phi)] = 0.$$

For the WT defined in this way one can easily prove that the action S is invariant:

$$D_\sigma^W(A, \gamma, \phi) S(A, \gamma, \phi) = 0. \quad (10)$$

As for the case of the DTs, it follows from (10) that the functional W is invariant in respect to the WTs:

$$D_\sigma^W(A, \gamma) W = 0 \quad (11)$$

where

$$D_\sigma^W(A, \gamma) \equiv D_\sigma^W(A, \gamma, 0) = \delta_\sigma^W \gamma_{\mu\nu} \frac{\delta}{\delta \gamma_{\mu\nu}} + \sum_i \delta_\sigma^W A^i \frac{\delta}{\delta A^i}. \quad (12)$$

The operators $D_\sigma^W(A, \gamma)$ form a representation of the WT algebra:

$$[D_\sigma^W(A, \gamma), D_\rho^W(A, \gamma)] = 0.$$

For a constant σ ,

$$\delta_\sigma^W \Phi_i = d_i \sigma \Phi_i, \quad M_j^i(\sigma) = \delta_j^i \sigma d_j, \quad \delta_\sigma^W A^i = \bar{d}_i \sigma A^i,$$

where the constant parameters $d_i = d_i(D)$, $\bar{d}_i \equiv D - d_i = \bar{d}_i(D)$ are the dimensions of the field Φ_i and source A^i . For the field $\Phi_0 \equiv \nabla_\mu \phi \nabla^\mu \phi$, $d_0 = D$ and for corresponding source A^0 , $\bar{d}_0 = 0$. If the source A^i is dimensionless, i.e. $\bar{d}_i(D) = 0$, for some definite value $D = \mathcal{D}_i$ of the space dimension, this dimension \mathcal{D}_i is called logarithmic for A_i . For given \mathcal{D} we denote the dimensions of fields and sources as $\bar{d}_j^{log} = \bar{d}_j|_{D=\mathcal{D}}$, $\bar{d}_j^{log} = \bar{d}_j|_{D=\mathcal{D}}$.

4. Renormalization

To perform the renormalization procedure we choose the source A that defines the logarithmic dimension of space \mathcal{D} which is considered as a fixed parameter specifying the renormalized theory. The generating functional of renormalized Green functions W_r is defined as follows:

$$W_r(J, \gamma) \equiv W(A(J, \gamma), \gamma).$$

The functions $A(J, \gamma)$ in the right hand side are of the form:

$$A^i = \mu^{\Delta_i} F^i(J, \gamma, D). \quad (13)$$

Here μ is an auxiliary scaling parameter ,

$$\Delta_i = \Delta_i(D) = \bar{d}_i(D) - \bar{d}_i^{\log}, \quad \left. \frac{\partial F^i(J, \gamma, D)}{\partial J^i} \right|_{J=0} = 1.$$

The function $F(J, \gamma, D)$ obeys the homogeneity condition

$$D^{\log}(J, \gamma) F^i(J, \gamma, D) = \bar{d}_i^{\log} F^i(J, \gamma, D), \quad (14)$$

where

$$D^{\log}(J, \gamma) \equiv \sum_i \bar{d}_i^{\log} J^i \frac{\delta}{\delta J^i} + 2\gamma_{\mu\nu} \frac{\delta}{\delta \gamma_{\mu\nu}}. \quad (15)$$

It is supposed also that the functions $J(A, \gamma)$ defined by (13) are the tensors with respect to the DTs:

$$\delta_\alpha^{DT} J^i = L_\alpha J^i. \quad (16)$$

The operators $D_\alpha^{DT}(A, \gamma)$, $D_\sigma^W(A, \gamma)$ can be represented in terms of the variables of W_r . It follows from (12), (16) that

$$D_\alpha^{DT}(A, \gamma) \equiv \mathcal{D}_\alpha^{DT}(J, \gamma) = D_\alpha^{DT}(J, \gamma), \quad (17)$$

$$D_\sigma^W(A, \gamma) \equiv \mathcal{D}_\sigma^W(J, \gamma) = \delta_\sigma^W \gamma_{\mu\nu} \frac{\delta}{\delta \gamma_{\mu\nu}} + \sum_i \delta_\sigma^W J^i \frac{\delta}{\delta J^i}. \quad (18)$$

The WT for the sources J can be obtained from (9), (12), (13):

$$\delta_\sigma^W J^i(J, \gamma) = \sum_j T_j^i \left\{ \sum_k [\sigma D \delta_k^j - M_k^j(\sigma)] F^k - 2\sigma \gamma_{\mu\nu} \frac{\delta}{\delta \gamma_{\mu\nu}} F^j \right\}.$$

Here T_j^i is the element of the matrix T defined as follows:

$$\sum_j T_j^i \frac{\partial F^j}{\partial J^k} = \delta_k^i.$$

Since the commutation relations are independent of the choose of variables,

$$[\mathcal{D}_\alpha^{DT}(J, \gamma), \mathcal{D}_\beta^{DT}(J, \gamma)] = \mathcal{D}_{[\alpha, \beta]}^{DT}(J, \gamma), \quad (19)$$

$$[\mathcal{D}_\sigma^W(J, \gamma), \mathcal{D}_\rho^W(J, \gamma)] = 0. \quad (20)$$

In virtue of (13), (14), (15), we obtain $D^{log}(J, \gamma) = D^{log}(A, \gamma)$. Hence, for constant σ it holds:

$$\begin{aligned} D_\sigma^W(A, \gamma) &= \sigma D^{log}(A, \gamma) + \sigma \sum_i \Delta_i A_i \frac{\delta}{\delta A_i} = \\ &= \sigma D^{log}(J, \gamma) + \sigma \sum_i \mu \frac{\partial A_i}{\partial \mu} \Big|_{J=const} \frac{\delta}{\delta A_i} = \\ &= \sigma D^{log}(J, \gamma) + \sigma \mu \frac{\partial}{\partial \mu} \Big|_{J=const} - \sigma \mu \frac{\partial}{\partial \mu} \Big|_{A=const}, \end{aligned}$$

and

$$\delta_\sigma^W J^i = D_\sigma^W(A, \gamma) J^i = \sigma \left(\bar{d}_i^{log} J^i - \mu \frac{\partial J^i}{\partial \mu} \Big|_{A=const} \right). \quad (21)$$

It follows from (4), (11) that $W_r(J, \gamma)$ is invariant in respect to the diffeomorphism and Weyl transformations:

$$\mathcal{D}_\alpha^{DT}(J, \gamma) W_r(J, \gamma) = 0, \quad (22)$$

$$\mathcal{D}_\sigma^W(J, \gamma) W_r(J, \gamma) = 0. \quad (23)$$

The functional $W_r(\lambda_r, J_r, J_\gamma)$ for usual models of the quantum field theory in Euclidean D -dimensional space can be constructed from $W_r(J, \gamma)$ in the following way:

$$W_r(\lambda_r, J_r, J_\gamma) = W_r(\lambda_r + J_r, \gamma^E + J_\gamma).$$

Here λ_r, J_r denote the set of renormalized parameters of the model and the set of the sources of renormalized composite operators. The source of the energy-momentum tensor and the metric of D -dimensional Euclidean space are denoted by J_γ, γ^E . If $\lambda_i \neq 0$ only in the case $\bar{d}_i^{log} \geq 0$, then the model is renormalizable. By appropriate choosing of the functions F^i in (13) for the renormalizable model, the functional $W_r(\lambda_r, J_r, J_\gamma)$ and the operators (17), (18) are finite for $J_i \rightarrow J_i + \gamma_i, \gamma \rightarrow \gamma^E + J_\gamma, D = \mathcal{D}$, finite parameters λ_r and sources J_r, J_γ [9], [10], [11]. For $W_r(\lambda_r, J_r, J_\gamma)$ the equation (23) by constant σ appears to be the usual renormalization group equation, if one takes into account (21) and (22) with $\alpha(x) = x\sigma$.

5. Critical point

Combining the DT and the WT one can obtain the transformation of the form:

$$\delta_\alpha = \delta_\alpha^{DT} + \delta_\sigma^W \Big|_{\sigma = \frac{\nabla_\alpha}{D}}.$$

As a consequence of the commutation relations (1), (6), it follows that

$$[\delta_\alpha, \delta_\beta] = \delta_{[\alpha, \beta]}.$$

This means, that the transformations δ_α form the representation of the diffeomorphism algebra. In virtue of (3), (5),

$$\delta_\alpha \gamma^{\mu\nu} = -\omega_\alpha^{\mu\nu}.$$

Hence, $\delta_\alpha \gamma^{\mu\nu} = 0$ for conformal α . Let us introduce the notations:

$$\mathcal{D}_\alpha(J, \gamma) \equiv \mathcal{D}_\alpha^{DT}(J, \gamma) + \mathcal{D}_\sigma^W(J, \gamma) |_{\sigma=\frac{\nabla\alpha}{D}},$$

$$\mathcal{D}_\alpha^r(\lambda_r, J_r, J_\gamma) \equiv \mathcal{D}_\alpha(\lambda_r + J_r, \gamma^E + J_\gamma).$$

It follows from (19), (20) that

$$[\mathcal{D}_\alpha^r(\lambda_r, J_r, J_\gamma), \mathcal{D}_\beta^r(\lambda_r, J_r, J_\gamma)] = \mathcal{D}_{[\alpha, \beta]}^r(\lambda_r J_r, J_\gamma),$$

i.e. the operators \mathcal{D}_α^r form the representation of the diffeomorphism algebra. In virtue of the diffeomorphism and Weyl invariance of W_r ,

$$\mathcal{D}_\alpha^r(\lambda_r, J_r, J_\gamma) W_r(\lambda_r, J_r, J_\gamma) = 0. \quad (24)$$

If

$$\mathcal{D}_\alpha^r(\lambda^*, 0, 0) = -\omega_\alpha^{\mu\nu}(\gamma^E) \frac{\delta}{\delta \gamma^{\mu\nu}}$$

for parameters $\lambda_r = \lambda^*$ of renormalized Euclidean theory, we call this set of parameters the critical point. It can be proved that this equality is equivalent to equalities defining the fixed point [11] in the renormalization group theory.

Let us denote $\mathcal{D}_\alpha^{conf}(J_r, J_\gamma) \equiv \mathcal{D}_\alpha^r(\lambda^*, J_r, J_\gamma)$ for conformal α . For the flat space, it follows from (24) that

$$\mathcal{D}_\alpha^{conf}(J_r, J_\gamma) W_r(\lambda^*, J_r, J_\gamma) = 0. \quad (25)$$

In virtue of $\mathcal{D}_\alpha^{conf}(0, 0) = 0$ the equality (25) means the common conformal invariance of Euclidean quantum field theory at the critical point.

6. Wilson expansion

The obtained infinite number of Ward identities presenting the investigated algebraic structure of conformal field theory are linear differential equations for W_r in variation derivatives of first order. The well known Wilson asymptotic expansion is written as a differential relation including variation derivatives of second order:

$$\frac{\delta}{\delta J^i(x)} \frac{\delta}{\delta J^k(y)} W_r = \sum_l \int dz K_{ijl}(x, y, z) \frac{\delta}{\delta J^l(z)} W_r. \quad (26)$$

It can be considered as a completing condition for considered algebraic structure. In conformal field theory the Wilson-expansion series are convergent [12], the functions $K_{ijl}(x, y, z)$ being 3-point correlation function are defined exactly up to finite number of constant (Wilson coefficients) by dimensions of field operators. In this paper we study restrictions following from (26) for dimensions of fields and Wilson coefficients. For this purpose we introduce some definitions and notations.

Let $x^{(n)}$ be a symmetric traceless tensor of rank n constructed from components x_a , $a = 1, \dots, d$, of D -dimensional vector x and Kronecker symbols:

$$x_{a_1 \dots a_n}^{(n)} = x_{a_1} \dots x_{a_n} - \text{traces}.$$

By definition, the contraction of $x^{(n)}$ with $y^{(n)}$ is written as

$$x^{(n)}y^{(n)} = \sum_{k=0}^{\{\frac{n}{2}\}} d_k^{(n)} x^{2k} y^{2k} (xy)^{n-2k} \equiv \mathcal{F}^{(n)}(x^2 y^2, xy)$$

where $\{n/2\}$ denotes the integer part of $n/2$, and $d_0^{(n)} \equiv 1$. With fixing $\mathcal{F}^{(n)}(0, b) = b^n$ and condition $\partial_x^2 \mathcal{F}^{(n)}(x^2 y^2, xy) = 0$ the function $\mathcal{F}^{(n)}(a, b)$ is defined unambiguously:

$$\mathcal{F}^{(n)}(a, b) = b^n \sum_{k=0}^{\{\frac{n}{2}\}} d_k^{(n)} \left(\frac{a}{b^2}\right)^k, \quad d_k^{(n)} = \frac{(-1)^k n! \Gamma(\xi + n - k - 1)}{4^k k! (n - 2k)! \Gamma(\xi + n - 1)}.$$

We shell use the following notations:

$$\begin{aligned} L(\alpha; x) &\equiv \frac{1}{(x^2)^\alpha}, \quad L^{(n)}(\alpha; x) \equiv x^{(n)} L(\alpha + n; x) \\ \lambda_a(x; y, z) &\equiv \frac{(x - y)_a}{(x - y)^2} - \frac{(x - z)_a}{(x - z)^2}, \\ V(\alpha_1, \alpha_2, \alpha_3; x_1, x_2, x_3) &= \\ &= L(\Delta_{12}; x_1 - x_2) L(\Delta_{13}; x_1 - x_3) L(\Delta_{23}; x_2 - x_3) = \\ &= (x_1 - x_2)^{-2\Delta_{12}} (x_1 - x_3)^{-2\Delta_{13}} (x_2 - x_3)^{-2\Delta_{23}} \\ V^{(n)}(\alpha, \beta, \gamma; x, y, z) &= V(\alpha, \beta, \gamma; x, y, z) \lambda^{(n)}(x; y, z). \end{aligned}$$

where x, y, z, x_1, x_2, x_3 are vectors of D -dimensional space, and

$$\Delta_{12} = \frac{\alpha_1 + \alpha_2 - \alpha_3}{2}, \quad \Delta_{13} = \frac{\alpha_1 + \alpha_3 - \alpha_2}{2}, \quad \Delta_{23} = \frac{\alpha_2 + \alpha_3 - \alpha_1}{2}.$$

The Wilson expansion for the 4-point correlation function $W(x, y, s, t)$ of the scalar field Φ_α with dimension α reads [3], [12] :

$$W(x, y, s, t) = \sum_{n,l} f_{ln} \int V^{(l)}(\beta_{ln}, \alpha, \alpha; z, x, y) V^{(l)}(\tilde{\beta}_{ln}, \alpha, \alpha; z, s, t) dz. \quad (27)$$

Here, f_{ln} are the Wilson coefficients, $V^{(l)}(z, x, y; \beta_{ln}, \alpha, \alpha)$ is (up to a constant amplitude) the 3-point correlation function of two fields Φ_α and one symmetric traceless l -component tensor field with dimension d_n , and $\tilde{\beta}_{ln}$ denotes the "shadow" in respect to d_{nl} dimension:

$$\tilde{\beta}_{ln} \equiv 2\xi - \beta_{ln} - 2l.$$

In virtue of $V^{(l)}(z, x, y; \beta_{ln}, \alpha, \alpha) = (-1)^l V^{(l)}(z, x, y; \beta_{ln}, \alpha, \alpha)$ and symmetry $W(x, y, s, t) = W(y, x, s, t) = W(x, s, y, t)$ we conclude that in (27) the summation parameter l is even, and the crossing symmetry equation

$$\begin{aligned} &\sum_{n,l} f_{ln} \int V^{(l)}(\beta_{ln}, \alpha, \alpha; z, x, y) V^{(l)}(\tilde{\beta}_{ln}, \alpha, \alpha; z, s, t) dz = \\ &= \sum_{n,l} f_{ln} \int V^{(l)}(\beta_{ln}, \alpha, \alpha; z, x, s) V^{(l)}(\tilde{\beta}_{ln}, \alpha, \alpha; z, y, t) dz \end{aligned} \quad (28)$$

must be fulfilled. It is a non-trivial restriction on the possible values of the Wilson coefficients and dimensions of fields. We show how one can be expressed in a form of exact analytical relations not containing coordinates x, y, s, t, z .

7. Crossing symmetry and sum rules

It will be convenient for compact writing of formulas to use the notation ξ for half dimension of space D and a short notation for the product of Γ -functions:

$$\xi \equiv \frac{D}{2}, \quad \Gamma(a, b, \dots, c) \equiv \Gamma(a)\Gamma(b) \cdots \Gamma(c).$$

If our expressions will contain the letter with prime, it will have the following meaning:

$$\alpha' \equiv \xi - \alpha.$$

The equality (28) is exact, but one is a integral equation with infinite number of terms, and a direct analysis of them is not easy. We obtain an evident form for the following consequence of crossing symmetry equation

$$\begin{aligned} & \sum_{n,l} f_{ln} \int \frac{(s-t)^{(m)} V^{(l)}(\beta_{ln}, \alpha, \alpha; z, x, y) V^{(l)}(\tilde{\beta}_{ln}, \alpha, \alpha; z, s, t)}{(s-t)^{2(\gamma+m)}} dx ds dz = \\ & = \sum_{n,l} f_{ln} \int \frac{(s-t)^{(m)} V^{(l)}(\beta_{ln}, \alpha, \alpha; z, x, s) V^{(l)}(\tilde{\beta}_{ln}, \alpha, \alpha; z, y, t)}{(s-t)^{2(\gamma+m)}} dx ds dz \end{aligned} \quad (29)$$

It is important that (29) must be fulfilled for arbitrary γ and all integer m .

The first step of calculation is a direct integration over x . In Appendix it is shown that with help of the formula

$$\begin{aligned} & \int L^{(n)}(\alpha; x-z) L(\beta; z-y) dz = \\ & = \pi^\xi \frac{\Gamma(\alpha', \beta', \xi - \alpha' - \beta' + n)}{\Gamma(\alpha + n, \beta, 2\xi - \alpha - \beta)} L^{(n)}(\alpha + \beta - \xi; x-y) \end{aligned} \quad (30)$$

one can integrate $V^{(l)}(z, x, y; \beta_{ln}, \alpha, \alpha)$ over x . After that the crossing symmetry equation takes the form:

$$\begin{aligned} & \sum_{n,l} f'_{ln} \int \frac{(s-t)^{(m)} (y-z)^{(l)} V^{(l)}(\tilde{\beta}_{ln}, \alpha, \alpha; z, s, t)}{(s-t)^{2(\gamma+m)} (y-z)^{2(\alpha+l-\xi)+\beta_{ln}}} ds dz = \\ & = \sum_{n,l} f'_{ln} \int \frac{(s-t)^{(m)} (s-z)^{(l)} V^{(l)}(\tilde{\beta}_{ln}, \alpha, \alpha; z, y, t)}{(s-t)^{2(\gamma+m)} (s-z)^{2(\alpha+l-\xi)+\beta_{ln}}} ds dz, \\ & f'_{ln} = f_{ln} \pi^\xi \frac{\Gamma(\alpha' + \beta_{ln}/2 + l, -\alpha', \xi - \beta_{ln}/2)}{\Gamma(\alpha - \beta_{ln}/2, 2\xi - \alpha, \beta_{ln}/2 + l)}. \end{aligned} \quad (31)$$

Now we do contractions of indexes in (31). For compact writing of results we use the shift operator T_ϵ acting on functions of ϵ as follows:

$$T_\epsilon f(\epsilon) = f(\epsilon + 1).$$

Let us denote $A = (\alpha_1, \alpha_2, \alpha_3)$, $X = (x_1, x_2, x_3)$, $E = (\epsilon_1, \epsilon_2, \epsilon_3)$, $R = (\rho_1, \rho_2, \rho_3)$,

$$S(A; X) \equiv S(\alpha_1, \alpha_2, \alpha_3; x_1, x_2, x_3) \equiv \quad (32)$$

$$\int L(\alpha_1; x_1 - y) L(\alpha_2; x_2 - y) L(\alpha_3; x_3 - y) dy,$$

$$S_n(A; X) \equiv S_n(\alpha_1, \alpha_2, \alpha_3; x_1, x_2, x_3) \equiv \quad (33)$$

$$\int L^{(n)}(\alpha_1; x_1 - y) \lambda^{(n)}(y; x_2, x_3) L(\alpha_2; x_2 - y) L(\alpha_3; x_3 - y) dy$$

It is shown in Appendix that the result of contraction of tensor indexes in the function $S_n(A; X)$ can be presented as

$$S_n(A; X) = T^{(n)}(E, R)G(A, E, R)S(A + R; X)|_{E=R=0}, \quad (34)$$

where

$$\begin{aligned} T^{(n)}(E, R) &= \mathcal{F}^{(n)}(\mathcal{M}(E, R), \mathcal{N}(E, R)), \\ \mathcal{M}(E, R) &= T_{\epsilon_1}^2 T_{\rho_1} [(T_{\epsilon_2} + T_{\epsilon_3})(T_{\epsilon_2} T_{\rho_2} + T_{\epsilon_3} T_{\rho_3}) - T_{\epsilon_2} T_{\epsilon_3} T_{\rho_1}], \\ \mathcal{N}(E, R) &= \frac{1}{2} T_{\epsilon_1} [(T_{\epsilon_2} + T_{\epsilon_3})(T_{\rho_2} - T_{\rho_3}) + (T_{\epsilon_2} - T_{\epsilon_3}) T_{\rho_1}], \\ G(A, E, R) &= \prod_{i=1}^3 \frac{\Gamma(\alpha_{nli} + \rho_i, 1 - \alpha'_{nli} + \rho_i)}{\Gamma(\alpha_{nli} + \epsilon_i, 1 - \alpha'_{nli})}, \\ \alpha_{nl1} &= \alpha + \frac{\beta_{nl}}{2} - \xi, \quad \alpha_{nl2} = \alpha_{nl3} = \xi - \frac{\beta_{nl}}{2} - l. \end{aligned}$$

With (34) the problem of integration in (31) is reduced to the case $l = 0$ and is solved directly by integration formula (30) (see Appendix). The final result is formulated in the following way. The crossing symmetry relation (31) is equivalent to the equality

$$\sum_{nl} f'_{nl} \Psi_{nl}(\alpha, \beta_{nl}; \gamma, m) = 0 \quad (35)$$

fulfilling for arbitrary value of parameter γ and integer $m \geq 0$. The function $\Psi_{nl}(\alpha, \beta_{nl}; \gamma, m)$ can be written in the form

$$\Psi_{nl}(\alpha, \beta_{nl}; \gamma, m) = T^{(n)}(E, R)Q(\alpha, \beta_{nl}, E, R)\Omega_m(\alpha, \beta_{nl}, \gamma, R)|_{E=R=0},$$

where

$$Q(\alpha, \beta_{nl}, E, R) = \frac{\pi^{2(\xi+1)}(-1)^{\rho_1+\rho_2}\Gamma(\alpha_{nl2} + \rho_3, 1 - \alpha'_{nl2} + \rho_3)}{\sin(\pi\alpha'_{nl1})\sin(\pi\alpha'_{nl2})\prod_{i=1}^3\Gamma(\alpha_{nli} + \epsilon_i, 1 - \alpha'_{nli})},$$

$$\begin{aligned} \Omega_m(\alpha, \beta_{nl}, \gamma, R) &= \\ &= \frac{\Gamma(\sigma'_1, \sigma'_2, \sigma'_3 + m, \sigma'_4 + m)}{\Gamma(\sigma_1 + m, \sigma_2 + m, \sigma_3, \sigma_4)} - \frac{\Gamma(\tau'_1, \tau'_2, \tau'_3 + m, \tau'_4 + m)}{\Gamma(\tau_1 + m, \tau_2 + m, \tau_3, \tau_4)}, \end{aligned}$$

with

$$\begin{aligned} \sigma_{nl1} &= \gamma + \alpha + \frac{\beta_{nl}}{2} - \xi + l, \quad \sigma_{nl2} = \gamma + \alpha - \frac{\beta_{nl}}{2} - l + \rho_2 + \rho_3, \\ \sigma_{nl3} &= 2\xi - \alpha - \rho_2 - \gamma, \quad \sigma_{nl4} = 3\xi + l - 2\alpha - \rho_1 - \rho_2 - \rho_3 - \gamma, \\ \tau_{nl1} &= \gamma, \quad \tau_{nl2} = \alpha - l + \rho_1 + \rho_3 + \gamma - \xi, \\ \tau_{nl3} &= 3\xi - \frac{\beta_{nl}}{2} - \alpha - \rho_1 - \gamma, \\ \tau_{nl4} &= 2\xi + 2l + \frac{\beta_{nl}}{2} - \alpha - \rho_1 - \rho_2 - \rho_3 - \gamma. \end{aligned}$$

8. Conclusion

It has been shown that for scalar Euclidean field theories at the critical point $\lambda_r = \lambda^*$ the operators $\mathcal{D}_\alpha^r(\lambda_r, J_r, J_\gamma)$ represent the generators of the DTs. The functional $W_r(\lambda_r, J_r, J_\gamma)$ is invariant in respect to the infinite set of the TDs defined by $\mathcal{D}_\alpha^r(\lambda_r, J_r, J_\gamma)$ and conformal transformations of D -dimensional Euclidean space. The Ward identities (24), (25) are the formal expressions of this invariance.

The Weyl invariance is described by equation (23), where the WT is presented with differential operator $\mathcal{D}_\sigma^W(J, \gamma)$. For constant σ it generates the usual renormalization group equation, if one puts in (24) $\alpha(x) = x\sigma$. With arbitrary $\sigma(x)$ (23) could be regarded as a solution of the considered in [14] problem of local generalization of renormalization group equations.

The Wilson expansion (26) is included as an additional relation for constructed algebraic structure. It was used for derivation of sum rule (35) which must be fulfilled for arbitrary λ and integer m . This nontrivial condition enables to hope that (35) contains essential information about dimensions of composite operators and Wilson coefficients.

We have obtained the following result. There is an infinite algebraic structure corresponding to each model of conformal field theory. It is given by the commutation relations (19), (20), Wilson expansion formula (26), Ward identities (23), (24), (25) and by choosing of logarithmic dimension \mathcal{D} defining dimensions of sources \vec{d}_i^{\log} . An addition restriction follows from the sum rules (35). Thus, the problem could be to elaborate the method of direct construction of the structure of such a kind.

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Appendix A. Details of calculations

We present some technical aspects of calculation methods used in this paper. The basic formula for our integration over coordinates of D -dimensional space is

$$\int L(\alpha; x - z)L(\beta; z - y)dz = v(\alpha, \beta, \gamma)L(\gamma'; x - y), \quad (\text{A.1})$$

$$\gamma = 2\xi - \alpha - \beta, \quad v(\alpha, \beta, \gamma) = \pi^\xi \frac{\Gamma(\alpha', \beta', \gamma')}{\Gamma(\alpha, \beta, \gamma)}$$

It can be easily proven with help of Fourier transformation [13]. For derivatives we have:

$$L^{(n)}(\alpha; x) = \frac{x^{(n)}}{x^{2(\alpha+n)}} = \frac{(-1)^n \Gamma(\alpha)}{2^n \Gamma(\alpha + n)} \partial_x^{(n)} \frac{1}{x^{2\alpha}} =$$

$$= \left(-\frac{T_\epsilon \partial_x}{2} \right)^{(n)} \frac{\Gamma(\alpha)}{\Gamma(\alpha + \epsilon) x^{2\alpha}} \Big|_{\epsilon=0} = \left(-\frac{T_\epsilon \partial_x}{2} \right)^{(n)} \frac{\Gamma(\alpha)}{\Gamma(\alpha + \epsilon)} L(\alpha; x) \Big|_{\epsilon=0}, \quad (\text{A.2})$$

$$\begin{aligned}
(\partial_x^2)^n L(\alpha; x) &= (\partial_x^2)^n \frac{1}{x^{2\alpha}} = 4^n \frac{\Gamma(\alpha + n, n + 1 - \alpha')}{\Gamma(\alpha, 1 - \alpha') x^{2(\alpha+n)}} = \\
&= (4T_\rho)^n \frac{\Gamma(\alpha + \rho, \rho + 1 - \alpha')}{\Gamma(\alpha, 1 - \alpha')} L(\alpha + \rho; x) \Big|_{\rho=0}.
\end{aligned} \tag{A.3}$$

From (A.1), (A.2) we obtain the generalization of (A.1):

$$\begin{aligned}
\int L^{(n)}(\alpha; x - z) L(\beta; z - y) dz &= v^{(n)}(\alpha, \beta, \gamma) L^{(n)}(\gamma'; x - y), \\
v^{(n)}(\alpha, \beta, \gamma) &\equiv \pi^\xi \frac{\Gamma(\alpha', \beta', \gamma' + n)}{\Gamma(\alpha + n, \beta, \gamma)}.
\end{aligned} \tag{A.4}$$

For calculation of the integral over x in (29) we use the inversion operator R acting on the function of the D -dimensional vectors and defined as

$$Rx \equiv \frac{x}{x^2}, \quad Rf(x, y, \dots, z) \equiv f(Rx, Ry, \dots, Rz).$$

One has the following properties

$$\begin{aligned}
R^2 &= 1, \quad R \frac{1}{x^{2\alpha}} = x^{2\alpha}, \quad R \frac{1}{(x - y)^{2\alpha}} = \frac{x^{2\alpha} y^{2\alpha}}{(x - y)^{2\alpha}}, \\
R\lambda(0; y, z) &= y - z, \quad \det \left(\frac{\partial(Rx)}{\partial x} \right) = \frac{1}{x^{2d}}.
\end{aligned}$$

Therefore, we obtain:

$$\begin{aligned}
\int V^{(l)}(0, x, y; \beta, \alpha, \alpha) dx &= R^2 \int V^{(l)}(0, x, y; \beta, \alpha, \alpha) dx = \\
&= R \int (x - y)^{(l)} V(0, Rx, Ry; \beta, \alpha, \alpha) d(Rx) = \\
&= R \int \frac{(x - y)^{(l)}}{x^{2(d-\alpha)} y^{-2\alpha} (x - y)^{2\alpha-\beta}} dx = \frac{v^{(l)}(\alpha - \beta/2 - l, 2\xi - \alpha, \beta/2 + l) y^{(l)}}{y^{2(\alpha-\xi)+\beta}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\int V^{(l)}(z, x, y; \beta, \alpha, \alpha) dx &= \\
v^{(l)}(\alpha - \beta/2 - l, 2\xi - \alpha, \beta/2 + l) L^{(l)}(\alpha - \xi + \frac{\beta}{2}; y - z).
\end{aligned}$$

Thus, after integration over x in (29) one obtains (31).

It follows from definition of $\lambda(x; y, z)_\mu$ and (A.2) that:

$$\begin{aligned}
\frac{\lambda(x; y, z)_\mu}{(x - y)^{2\alpha} (x - z)^{2\beta}} &= \\
&= D_\mu(y, z; \epsilon_1, \epsilon_2) \frac{\Gamma(\alpha, \beta)}{\Gamma(\alpha + \epsilon_1, \beta + \epsilon_2) (x - y)^{2\alpha} (x - z)^{2\beta}} \Big|_{\epsilon_1=\epsilon_2=0},
\end{aligned}$$

and

$$\begin{aligned}
\frac{\lambda^{(n)}(x; y, z)}{(x - y)^{2\alpha} (x - z)^{2\beta}} &= \\
&= D^{(n)}(y, z; \epsilon_1, \epsilon_2) \frac{\Gamma(\alpha, \beta)}{\Gamma(\alpha + \epsilon_1, \beta + \epsilon_2) (x - y)^{2\alpha} (x - z)^{2\beta}} \Big|_{\epsilon_1=\epsilon_2=0},
\end{aligned}$$

where

$$D_\mu(y, z; \epsilon_1, \epsilon_2) \equiv \frac{1}{2} \left(T_{\epsilon_1} \frac{\partial}{\partial y_\mu} - T_{\epsilon_2} \frac{\partial}{\partial z_\mu} \right).$$

Using (A.2) and notations (32), (33) we obtain the following equality

$$S_n(A; X) = \left(-\frac{T_{\epsilon_1} \partial_{x_1}}{2} \right)^{(n)} D^{(n)}(x_2, x_3; \epsilon_2, \epsilon_3) G(A, E) S(A; X)|_{E=0},$$

$$G(A, E) = \frac{\Gamma(\alpha_1, \alpha_2, \alpha_3)}{\Gamma(\alpha_1 + \epsilon_1, \alpha_2 + \epsilon_2, \alpha_3 + \epsilon_3)}$$

The function $S(A; X)$ is invariant in respect to translations, i.e. $S(A; x_1, x_2, x_3) = S(A; x_1 + y, x_2 + y, x_3 + y)$. Therefore

$$(\partial_{x_1} + \partial_{x_2} + \partial_{x_3}) S(A; X) = 0,$$

$$\partial_{x_i} \partial_{x_j} S(A; X) = \frac{1}{2} (\partial_{x_k}^2 - \partial_{x_i}^2 - \partial_{x_j}^2) S(A; X),$$

where $i, j, k = 1, 2, 3$ and $i \neq j, i \neq k, j \neq k$. By means of (A.3), we obtain

$$\left(\frac{T_{\epsilon_1} \partial_{x_1}}{2} \right)^{2k} D^{2k}(x_2, x_3; \epsilon_2, \epsilon_3) G(A, E) S(A; X)|_{E=0} =$$

$$= \frac{1}{4^{2k}} T_{\epsilon_1}^{2k} \partial_{x_1}^{2k} (T_{\epsilon_2}^2 \partial_{x_2}^2 + T_{\epsilon_3}^2 \partial_{x_3}^2 - 2T_{\epsilon_2} T_{\epsilon_3} \partial_{x_2} \partial_{x_3})^k G(A, E) S(A; X)|_{E=0} =$$

$$= \mathcal{M}(E, R)^k G(A, E, R) S(A + R; X)|_{E=0, R=0},$$

where $R = (\rho_1, \rho_2, \rho_3)$,

$$\mathcal{M}(E, R) = T_{\epsilon_1}^2 T_{\rho_1} [(T_{\epsilon_2} + T_{\epsilon_3})(T_{\epsilon_2} T_{\rho_2} + T_{\epsilon_3} T_{\rho_3}) - T_{\epsilon_2} T_{\epsilon_3} T_{\rho_1}],$$

$$G(A, E, R) = \prod_{i=1}^3 \frac{\Gamma(\alpha_i + \rho_i, 1 - \alpha'_i + \rho_i)}{\Gamma(\alpha_i + \epsilon_i, 1 - \alpha'_i)}.$$

Analogously we obtain the relation

$$\left(-\frac{T_{\epsilon_1} \partial_{x_1}}{2} D(x_2, x_3; \epsilon_2, \epsilon_3) \right)^l G(A, E) S(A; X)|_{E=0} =$$

$$= \frac{1}{4^l} T_{\epsilon_1}^l (T_{\epsilon_3} \partial_{x_1} \partial_{x_3} - T_{\epsilon_2} \partial_{x_1} \partial_{x_2})^l G(A, E) S(A; X)|_{E=0} =$$

$$= \mathcal{N}(E, R)^l G(A, E, R) S(A + R; X)|_{E=0, R=0},$$

where

$$\mathcal{N}(E, R) = \frac{1}{2} T_{\epsilon_1} [(T_{\epsilon_2} + T_{\epsilon_3})(T_{\rho_2} - T_{\rho_3}) + (T_{\epsilon_2} - T_{\epsilon_3})T_{\rho_1}].$$

Thus, we have shown that the result of contraction of tensor indexes in the function $S_n(A; X)$ can be presented as

$$S_n(A; X) = T^{(n)}(E, R) G(A, E, R) S(A + R; X)|_{E=R=0},$$

where

$$T^{(n)}(E, R) = \mathcal{F}^{(n)}(\mathcal{M}(E, R), \mathcal{N}(E, R)).$$

We can write the crossing symmetry equation (31) as

$$\begin{aligned} & \sum_{n,l} f'_{nl} \int S_n(\alpha_{nl1}, \alpha_{nl2}, \alpha_{nl3}; y, s, t) L^{(m)}(\zeta_{nl} + \gamma; s - t) ds = \\ & = \sum_{n,l} f'_{nl} \int S_n(\alpha_{nl1}, \alpha_{nl2}, \alpha_{nl3}; s, y, t) L^{(m)}(\gamma; s - t) L(\zeta_{nl}; y - t) ds \end{aligned}$$

with

$$\alpha_{nl1} = \frac{\beta_{nl}}{2} - \alpha', \quad \alpha_{nl2} = \alpha_{nl3} = \frac{\tilde{\beta}_{nl}}{2}, \quad \zeta_{nl} = \alpha - \frac{\tilde{\beta}_{nl}}{2}.$$

By means of (A.4) we obtain

$$\begin{aligned} & \int S(\alpha_1, \alpha_2, \alpha_3; y, s, t) L^{(m)}(\zeta + \gamma; s - t) ds = \\ & = \Phi_1(\alpha_1, \alpha_2, \alpha_3, \zeta, \gamma, m) L^{(m)}(\alpha_1 + \alpha_2 + \alpha_3 + \zeta + \gamma - 2\xi; t - y), \\ & \int S(\alpha_1, \alpha_2, \alpha_3; s, y, t) L^{(m)}(\gamma; s - t) L(\zeta; s - y) ds = \\ & = \Phi_2(\alpha_1, \alpha_2, \alpha_3, \zeta, \gamma, m) L^{(m)}(\alpha_1 + \alpha_2 + \alpha_3 + \zeta + \gamma - 2\xi; t - y), \end{aligned}$$

where

$$\begin{aligned} \Phi_1(\alpha_1, \alpha_2, \alpha_3, \zeta, \gamma, m) &= v_m(\gamma + \zeta, \alpha_2, 2\xi - \alpha_2 - \gamma - \zeta) \times \\ &\times v_m(\gamma + \zeta + \alpha_2 + \alpha_3 - \xi, \alpha_1, 3\xi - \alpha_1 - \alpha_2 - \alpha_3 - \gamma - \zeta), \\ \Phi_2(\alpha_1, \alpha_2, \alpha_3, \zeta, \gamma, m) &= v_m(\gamma, \alpha_1, 2\xi - \alpha_1 - \gamma) \times \\ &\times v_m(\alpha_1 + \alpha_2 + \gamma - \xi, \alpha_3, 3\xi - \alpha_1 - \alpha_2 - \alpha_3 - \gamma). \end{aligned}$$

Let us denote

$$\begin{aligned} \Phi(A, \zeta, \gamma, m) &= \Phi(\alpha_1, \alpha_2, \alpha_3, \zeta, \gamma, m) \equiv \Phi_1(A, \zeta, \gamma, m) - \Phi_2(A, \zeta, \gamma, m), \\ \Psi_{nl}(\alpha, \beta_{nl}; \gamma, m) &= T^{(n)}(E, R) G(A_{nl}, E, R) \Phi(A_{nl} + R, \gamma, \zeta_{nl})|_{E=R=0}, \end{aligned}$$

where $A_{nl} \equiv (\alpha_{nl1}, \alpha_{nl2}, \alpha_{nl3})$. The equation (31) reads

$$\sum_{nl} f'_{nl} \Psi_{nl}(\alpha, \beta_{nl}; \gamma, m) = 0.$$

It is fulfilled for arbitrary value of parameter γ and integer $m \geq 0$. Using notations $\bar{E} \equiv (\epsilon_1, \epsilon_3, \epsilon_2)$, $\bar{R} \equiv (\rho_1, \rho_3, \rho_2)$,

$$\mathcal{A}_m(\nu_1, \nu_2; \nu_3, \nu_4) \equiv \frac{\Gamma(\nu'_1, \nu'_2, \nu'_3 + m, \nu'_4 + m)}{\Gamma(\nu_1 + m, \nu_2 + m, \nu_3, \nu_4)},$$

we write

$$\begin{aligned} \Phi_1(A_{nl} + R, \gamma, m) &= \pi^{2\xi} \prod_{i=1}^2 \frac{\Gamma(\alpha'_{nli} - \rho_i)}{\Gamma(\alpha_{nli} + \rho_i)} \mathcal{A}_m(\sigma_{nl1}, \sigma_{nl2}; \sigma_{nl3}, \sigma_{nl4}), \\ \sigma_{nl1} &= \gamma + \alpha + \frac{\beta_{nl}}{2} - \xi + l, \quad \sigma_{nl2} = \gamma + \alpha - \frac{\beta_{nl}}{2} - l + \rho_2 + \rho_3, \\ \sigma_{nl3} &= 2\xi - \alpha - \rho_2 - \gamma, \quad \sigma_{nl4} = 3\xi + l - 2\alpha - \rho_1 - \rho_2 - \rho_3 - \gamma, \\ \Phi_2(A_{nl} + \bar{R}, \gamma, m) &= \pi^{2\xi} \prod_{i=1}^2 \frac{\Gamma(\alpha'_{nli} - \rho_i)}{\Gamma(\alpha_{nli} + \rho_i)} \mathcal{A}_m(\tau_{nl1}, \tau_{nl2}; \tau_{nl3}, \tau_{nl4}), \end{aligned}$$

$$\tau_{nl1} = \gamma, \quad \tau_{nl2} = \alpha - l + \rho_1 + \rho_3 + \gamma - \xi,$$

$$\tau_{nl3} = 3\xi - \frac{\beta_{nl}}{2} - \alpha - \rho_1 - \gamma,$$

$$\tau_{nl4} = 2\xi + 2l + \frac{\beta_{nl}}{2} - \alpha - \rho_1 - \rho_2 - \rho_3 - \gamma.$$

Taking into account that $\Gamma(1-x, x) = \pi/\sin(\pi x)$, we obtain

$$\begin{aligned} G(A, E, R,) \prod_{i=1}^2 \frac{\Gamma(\alpha'_{nli} - \rho_i)}{\Gamma(\alpha_{nli} + \rho_i)} &= \\ &= \prod_{i=1}^2 \frac{\Gamma(\alpha_{nli} + \rho_i, 1 - \alpha'_{nli} + \rho_i, \alpha'_{nli} - \rho_i)}{\Gamma(\alpha_{nli} + \epsilon_i, 1 - \alpha'_{nli}, \alpha_{nli} + \rho_i)} \frac{\Gamma(\alpha_{nl2} + \rho_3, 1 - \alpha'_{nl2} + \rho_3)}{\Gamma(\alpha_{nl2} + \epsilon_3, 1 - \alpha'_{nl2})} = \\ &= \pi^2 \frac{\Gamma(\alpha_{nl2} + \rho_3, 1 - \alpha'_{nl2} + \rho_3)}{\sin(\pi(\alpha'_{nl1} - \rho_1)) \sin(\pi(\alpha'_{nl2} - \rho_2))} \prod_{i=1}^3 \frac{1}{\Gamma(\alpha_{nli} + \epsilon_i, 1 - \alpha'_{nli})}. \end{aligned}$$

For integer m , and for even n

$$\begin{aligned} \sin(\alpha + \pi m) &= (-1)^m \sin(\alpha), \\ T^{(n)}(E, R)G(A_{nl}, E, R)\Phi_2(A_{nl} + R, \gamma, \zeta_{nl})|_{E=R=0} &= \\ &= T^{(n)}(E, R)G(A_{nl}, \bar{E}, \bar{R})\Phi_2(A_{nl} + \bar{R}, \gamma, \zeta_{nl})|_{E=R=0} = \\ &= T^{(n)}(E, R)G(A_{nl}, E, R)\Phi_2(A_{nl} + \bar{R}, \gamma, \zeta_{nl})|_{E=R=0}. \end{aligned}$$

Therefore we can present the function $\Psi_{nl}(\alpha, \beta_{nl}; \gamma, m)$ as follows:

$$\Psi_{nl}(\alpha, \beta_{nl}; \gamma, m) = T^{(n)}(E, R)Q(\alpha, \beta_{nl}, E, R)\Omega_m(\alpha, \beta_{nl}, \gamma, R)|_{E=R=0},$$

where

$$\begin{aligned} Q(\alpha, \beta_{nl}, E, R) &= \frac{\pi^{2(\xi+1)}(-1)^{\rho_1+\rho_2}\Gamma(\alpha_{nl2} + \rho_3, 1 - \alpha'_{nl2} + \rho_3)}{\sin(\pi\alpha'_{nl1}) \sin(\pi\alpha'_{nl2}) \prod_{i=1}^3 \Gamma(\alpha_{nli} + \epsilon_i, 1 - \alpha'_{nli})}, \\ \Omega_m(\alpha, \beta_{nl}, \gamma, R) &= \\ &= \mathcal{A}_m(\sigma_{nl1}, \sigma_{nl2}; \sigma_{nl3}, \sigma_{nl4}) - \mathcal{A}_m(\tau_{nl1}, \tau_{nl2}; \tau_{nl3}, \tau_{nl4}). \end{aligned}$$

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